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On the coexistence of position and momentum observables

Claudio Carmeli[1], **Teiko Heinonen**[2] **and Alessandro Toigo**[1]

[1] Dipartimento di Fisica, Università di Genova, and INFN, Sezione di Genova, Via Dodecaneso 33, 16146 Genova, Italy
[2] Department of Physics, University of Turku, FIN-20014 Turku, Finland

E-mail: carmeli@ge.infn.it, teiko.heinonen@utu.fi and toigo@ge.infn.it

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Abstract
We investigate the problem of coexistence of position and momentum observables. We characterize those pairs of position and momentum observables which have a joint observable.

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1. Introduction

The problem of joint measurability of position and momentum observables in quantum mechanics has a long history and different viewpoints have been presented (see, e.g., [1]). According to a common view sharp position and momentum observables are complementary quantities and therefore are not jointly measurable. This is also illustrated, for instance, by the fact that the Wigner distribution is not a probability distribution. The advent of positive operator measures in quantum mechanics has made further mathematically sound development possible. In this framework certain observables, which are interpreted as unsharp position and momentum observables, have joint measurements [2–4].

Although a collection of important results has been obtained, it seems that the fundamental problem of joint measurability of position and momentum observables has not yet been solved in its full generality. Coexistent position and momentum observables (in the sense of Ludwig [5]) have not been characterized so far.

Our analysis of this problem proceeds in the following way. In section 2 we fix the notation and recall some concepts which are essential for our investigation. In section 3 position and momentum observables are defined through their behaviour under the appropriate symmetry transformations. In section 4 we follow a recent work of Werner [6] to characterize those pairs of position and momentum observables which are functionally coexistent and can thus be measured jointly. Also some properties of joint observables are investigated.

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In section 5 we present a few observations on the general problem of coexistence of position and momentum observables.

2. Coexistence and joint observables

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. The null operator and the identity operator are denoted by O and I, respectively. Let Ω be a (nonempty) set and $\mathcal{A}$ a σ-algebra of subsets of Ω. A set function $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is an *operator measure* if it is σ-additive (with respect to the weak operator topology). If $E(X) \geqslant O$ for all $X \in \mathcal{A}$, we say that E is *positive*, and E is *normalized* if $E(\Omega) = I$. The range of an operator measure E is denoted by ran(E), that is,

$$\mathrm{ran}(E) = \{E(X) \mid X \in \mathcal{A}\}.$$

In quantum mechanics *observables* are represented as normalized positive operator measures and *states* as positive operators of trace one. We denote by $\mathcal{S}(\mathcal{H})$ the set of states. For an observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and a state $T \in \mathcal{S}(\mathcal{H})$, we let p_T^E denote the probability measure on Ω, defined by

$$p_T^E(X) = \mathrm{tr}[TE(X)], \qquad X \in \mathcal{A}.$$

This is the probability distribution of the measurement outcomes when the system is in the state T and the observable E is measured. If the range of E contains only projections, then E is called a *sharp observable*. For more about observables as normalized positive operator measures, the reader may refer to the monographs [4, 7–9].

The notions of coexistence, functional coexistence and joint observables are essential when the joint measurability of quantum observables is analysed. We next briefly recall the definitions of these concepts. For further details we refer to a convenient survey [10] and references given therein.

Definition 1. *Let $(\Omega_i, \mathcal{A}_i)$, $i = 1, 2$, be measurable spaces and let $E_i : \mathcal{A}_i \to \mathcal{L}(\mathcal{H})$ be observables.*

(i) E_1 and E_2 are coexistent *if there is a measurable space $(\Omega, \mathcal{A})$ and an observable $G : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that*

$$\mathrm{ran}(E_1) \cup \mathrm{ran}(E_2) \subseteq \mathrm{ran}(G).$$

(ii) E_1 and E_2 are functionally coexistent *if there is a measurable space $(\Omega, \mathcal{A})$, an observable $G : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and measurable functions $f_1 : \Omega \to \Omega_1$, $f_2 : \Omega \to \Omega_2$, such that for any $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$,*

$$E_1(X) = G\big(f_1^{-1}(X)\big), \qquad E_2(Y) = G\big(f_2^{-1}(Y)\big).$$

Functionally coexistent observables are coexistent, but it is an open question if the reverse holds. We now confine our discussion to observables on $\mathbb{R}$. We denote by $\mathcal{B}(\mathbb{R}^n)$ the Borel σ-algebra of $\mathbb{R}^n$.

Definition 2. *Let $E_1, E_2 : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ be observables. An observable $G : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ is their* joint observable *if for all $X, Y \in \mathcal{B}(\mathbb{R})$,*

$$E_1(X) = G(X \times \mathbb{R}), \qquad E_2(Y) = G(\mathbb{R} \times Y).$$

In this case E_1 and E_2 are the margins *of G.*

For observables E_1 and E_2 defined on $\mathcal{B}(\mathbb{R})$ the existence of a joint observable is equivalent to their functional coexistence. These conditions are also equivalent to the *joint measurability* of E_1 and E_2 in the sense of the quantum measurement theory (see [10], section 7).

The *commutation domain* of observables E_1 and E_2, denoted by $\mathrm{com}(E_1, E_2)$, is the closed subspace of $\mathcal{H}$ defined as

$$\mathrm{com}(E_1, E_2) = \{\psi \in \mathcal{H} \mid E_1(X)E_2(Y)\psi - E_2(Y)E_1(X)\psi = 0 \ \forall X, Y \in \mathcal{B}(\mathbb{R})\}.$$

If E_1 and E_2 are sharp observables, then E_1 and E_2 are coexistent only if they are functionally coexistent and this is the case exactly when $\mathrm{com}(E_1, E_2) = \mathcal{H}$. In general, for two observables E_1 and E_2 the condition $\mathrm{com}(E_1, E_2) = \mathcal{H}$ is sufficient but not necessary for the functional coexistence of E_1 and E_2.

In conclusion, given a pair of observables one may pose the questions of their commutativity, functional coexistence and coexistence, in the order of increasing generality.

3. Position and momentum observables

Let us shortly recall the standard description of a spin-0 particle in the one-dimensional space $\mathbb{R}$. Fix $\mathcal{H} = L^2(\mathbb{R})$ and let U and V be the one-parameter unitary representations on $\mathcal{H}$, acting on $\psi \in \mathcal{H}$ as

$$[U(q)\psi](x) = \psi(x - q), \qquad [V(p)\psi](x) = \mathrm{e}^{\mathrm{i}px}\psi(x).$$

The representations U and V correspond to space translations and velocity boosts. They can be combined to form the following irreducible projective representation W of $\mathbb{R}^2$:

$$W(q, p) = \mathrm{e}^{\mathrm{i}qp/2}U(q)V(p). \tag{1}$$

Let P and Q be the self-adjoint operators generating U and V, that is, $U(q) = \mathrm{e}^{-\mathrm{i}qP}$ and $V(p) = \mathrm{e}^{\mathrm{i}pQ}$ for every $q, p \in \mathbb{R}$. We denote by Π_P and Π_Q the sharp observables corresponding to the operators P and Q. For any $X \in \mathcal{B}(\mathbb{R})$ and $\psi \in \mathcal{H}$ we then have

$$\Pi_Q(X)\psi = \chi_X\psi, \qquad \Pi_P(X) = \mathcal{F}^{-1}\Pi_Q(X)\mathcal{F}, \tag{2}$$

where χ_X is the characteristic function of X, and $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is the Fourier–Plancherel operator. Sharp observables Π_Q and Π_P correspond to position and momentum measurements of absolute precision. We call them *the canonical position observable* and *the canonical momentum observable*, respectively.

We take the symmetry properties of Π_Q and Π_P as the defining properties of generic position and momentum observables. An observable $E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ is a *position observable* if, for all $q, p \in \mathbb{R}$ and $X \in \mathcal{B}(\mathbb{R})$,

$$U(q)E(X)U(q)^* = E(X + q), \tag{3}$$

$$V(p)E(X)V(p)^* = E(X). \tag{4}$$

This means that a position observable is defined as a translation covariant and velocity boost invariant observable. In our previous paper [11] we have shown that these conditions are satisfied exactly when there is a probability measure $\rho : \mathcal{B}(\mathbb{R}) \to [0, 1]$ such that

$$E(X) = E_\rho(X) := \int \rho(X - q)\,\mathrm{d}\Pi_Q(q), \qquad X \in \mathcal{B}(\mathbb{R}), \tag{5}$$

where $X - q = \{x - q \mid x \in X\}$. A position observable E_ρ can be interpreted as a *fuzzy version* of the canonical position observable Π_Q, unsharpness being characterized by the probability measure ρ (see, e.g., [12–14]). We call E_ρ a *fuzzy position observable* if E_ρ is not a sharp observable. We denote by $M(\mathbb{R})$ the set of complex measures on $\mathbb{R}$ and $M_1^+(\mathbb{R})$ is the subset of probability measures. For any $\lambda \in M(\mathbb{R})$, $\widehat{\lambda}$ denotes the Fourier–Stieltjes transform of λ.

Proposition 3. *Let $\rho_1, \rho_2 \in M_1^+(\mathbb{R})$, $\rho_1 \neq \rho_2$. Then $E_{\rho_1} \neq E_{\rho_2}$.*

Proof. For $\psi \in \mathcal{H}$, we define the real measure λ_ψ by

$$\lambda_\psi(X) = \langle \psi | (E_{\rho_1}(X) - E_{\rho_2}(X)) \psi \rangle = \mu_\psi * (\rho_1 - \rho_2)(X),$$

where $*$ is the convolution and $\mathrm{d}\mu_\psi(x) = |\psi(x)|^2\,\mathrm{d}x$. Taking the Fourier transform we get

$$\widehat{\lambda}_\psi = \widehat{\mu}_\psi \cdot (\widehat{\rho_1} - \widehat{\rho_2}),$$

where $\widehat{\lambda}_\psi, \widehat{\mu}_\psi, \widehat{\rho_1}$ and $\widehat{\rho_2}$ are continuous functions. By injectivity of the Fourier–Stieltjes transform we have $\widehat{\rho_1} \neq \widehat{\rho_2}$. Thus, choosing ψ such that $\widehat{|\psi|^2}(p) \neq 0$ for every $p \in \mathbb{R}$, we have $\widehat{\lambda}_\psi \neq 0$. This means that $\lambda_\psi \neq 0$ and hence, $E_{\rho_1} \neq E_{\rho_2}$. □

By proposition 3 there is one-to-one correspondence between the set of position observables and $M_1^+(\mathbb{R})$. A position observable E_ρ is a sharp observable if and only if $\rho = \delta_x$ for some $x \in \mathbb{R}$, where δ_x is the Dirac measure concentrated at x [11]. Since the Dirac measures are the extreme elements of the convex set $M_1^+(\mathbb{R})$, the sharp position observables are the extreme elements of the set of position observables. The canonical position observable Π_Q corresponds to the Dirac measure δ_0.

In an analogous way, a momentum observable is defined as a velocity boost covariant and translation invariant observable. Thus, an observable $F : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ is a *momentum observable* if, for all $q, p \in \mathbb{R}$ and $Y \in \mathcal{B}(\mathbb{R})$,

$$V(p)F(Y)V(p)^* = F(Y + p), \tag{6}$$

$$U(q)F(Y)U(q)^* = F(Y). \tag{7}$$

Since an observable E is a position observable if and only if $\mathcal{F}^{-1}E\mathcal{F}$ is a momentum observable, the previous discussion on position observables is easily converted to the case of momentum observables. In particular, an observable F satisfies conditions (6) and (7) if and only if there is a probability measure $\nu : \mathcal{B}(\mathbb{R}) \to [0, 1]$ such that $F = F_\nu$, where

$$F_\nu(Y) := \int \nu(Y - p)\,\mathrm{d}\Pi_P(p), \qquad Y \in \mathcal{B}(\mathbb{R}). \tag{8}$$

For completeness we give a proof of the following known fact [15], which will be needed later.

Proposition 4. *A position observable E_ρ and a momentum observable F_ν are totally noncommutative, that is,* $\mathrm{com}(E_\rho, F_\nu) = \{0\}$.

Proof. It is shown in [16, 17] that for functions $f, g \in L^\infty(\mathbb{R})$ the equation

$$f(Q)g(P) - g(P)f(Q) = O$$

holds if and only if one of the following is satisfied: (i) either $f(Q)$ or $g(P)$ is a multiple of the identity operator, (ii) f and g are both periodic with minimal periods a, b satisfying $2\pi/ab \in \mathbb{Z}\backslash\{0\}$.

Let $X \subset \mathbb{R}$ be a bounded interval. Then the operators $E_\rho(X)$ and $F_\nu(X)$ are not multiples of the identity operator. Indeed, let us assume, in contrast, that $E_\rho(X) = cI$ for some $c \in \mathbb{R}$.

Denote $a = 2|X|$, where $|X|$ is the length of X. Then the sets $X + na$, $n \in \mathbb{Z}$, are pairwisely disjoint and

$$I \geqslant E_\rho(\cup_{n\in\mathbb{Z}}(X+na)) = \sum_{n=-\infty}^{\infty} E_\rho(X+na)$$
$$= \sum_{n=-\infty}^{\infty} U(na)E_\rho(X)U(na)^* = \sum_{n=-\infty}^{\infty} cI.$$

This means that $c = 0$. However, since $|X| > 0$, we have $E_\rho(X) \neq O$ (see, e.g., [18]). Thus,

$$O \neq E_\rho(X) = cI = O,$$

and $E_\rho(X)$ is not a multiple of the identity operator. Moreover, since $\rho(\mathbb{R}) = 1$, the function $q \mapsto \rho(X - q)$ is not periodic. We conclude that, by the above-mentioned result, the operators $E_\rho(X)$ and $F_\nu(X)$ do not commute and hence, $\mathrm{com}(E_\rho, F_\nu) \neq \mathcal{H}$.

Assume then that there exists $\psi \neq 0$, $\psi \in \mathrm{com}(E_\rho, F_\nu)$. Using the symmetry properties (3), (4), (6) and (7), a short calculation shows that for any $q, p \in \mathbb{R}$, $U(q)V(p)\psi \in \mathrm{com}(E_\rho, F_\nu)$. This implies that $\mathrm{com}(E_\rho, F_\nu)$ is invariant under the irreducible projective representation W defined in (1). As $\mathrm{com}(E_\rho, F_\nu)$ is a closed subspace of $\mathcal{H}$, it follows that either $\mathrm{com}(E_\rho, F_\nu) = \{0\}$ or $\mathrm{com}(E_\rho, F_\nu) = \mathcal{H}$. Since the latter possibility is ruled out, this completes the proof. □

4. Joint observables of position and momentum observables

Looking at the symmetry conditions (3), (4), (6) and (7), and equation (1), it is clear that an observable $G : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ has a position observable and a momentum observable as its margins if and only if, for all $q, p \in \mathbb{R}$ and $X, Y \in \mathcal{B}(\mathbb{R})$, the following conditions hold:

$$W(q,p)G(X\times\mathbb{R})W(q,p)^* = G(X\times\mathbb{R} + (q,p)), \tag{9}$$
$$W(q,p)G(\mathbb{R}\times Y)W(q,p)^* = G(\mathbb{R}\times Y + (q,p)). \tag{10}$$

Definition 5. *An observable $G : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ is a covariant phase-space observable if for all $q, p \in \mathbb{R}$ and $Z \in \mathcal{B}(\mathbb{R}^2)$,*

$$W(q,p)G(Z)W(q,p)^* = G(Z + (q,p)). \tag{11}$$

It is trivial that (11) implies (9) and (10) and, hence, a covariant phase-space observable is a joint observable of some position and momentum observables. To our knowledge, it is an open question whether (9) and (10) imply (11).

For any $T \in \mathcal{S}(\mathcal{H})$, we define an observable $G_T : \mathcal{B}(\mathbb{R}^2) \to \mathcal{L}(\mathcal{H})$ by

$$G_T(Z) = \frac{1}{2\pi}\int_Z W(q,p)TW(q,p)^*\,\mathrm{d}q\,\mathrm{d}p, \qquad Z \in \mathcal{B}(\mathbb{R}^2). \tag{12}$$

The observable G_T is a covariant phase-space observable. Moreover, if G is a covariant phase-space observable, then $G = G_T$ for some state $T \in \mathcal{S}(\mathcal{H})$ [4, 19, 20].

Proposition 6. *Let $T_1, T_2 \in \mathcal{S}(\mathcal{H})$, $T_1 \neq T_2$. Then $G_{T_1} \neq G_{T_2}$.*

Proof. Let us first note that for any $T, S \in \mathcal{S}(\mathcal{H})$ and $Z \in \mathcal{B}(\mathbb{R}^2)$,

$$\begin{aligned} p_S^{G_T}(Z) &= \frac{1}{2\pi}\int_Z \operatorname{tr}[SW(q,p)TW(q,p)^*]\,\mathrm{d}q\,\mathrm{d}p \\ &= \frac{1}{2\pi}\int_Z \operatorname{tr}[TW(q,p)^*SW(q,p)]\,\mathrm{d}q\,\mathrm{d}p \\ &= \frac{1}{2\pi}\int_{-Z} \operatorname{tr}[TW(q,p)SW(q,p)^*]\,\mathrm{d}q\,\mathrm{d}p \\ &= p_T^{G_S}(-Z). \end{aligned}$$

Let $T_1, T_2 \in \mathcal{S}(\mathcal{H})$ and assume that $G_{T_1} = G_{T_2}$. This means that for any $S \in \mathcal{S}(\mathcal{H})$,

$$p_S^{G_{T_1}} = p_S^{G_{T_2}}, \tag{13}$$

which is, by the previous observation, equivalent to

$$p_{T_1}^{G_S} = p_{T_2}^{G_S}. \tag{14}$$

Let S be a state such that G_S is an informationally complete observable (see [21]). Then (14) implies that $T_1 = T_2$. □

Let G_T be a covariant phase-space observable and let $\sum_i \lambda_i|\varphi_i\rangle\langle\varphi_i|$ be the spectral decomposition of the state T. The margins of G_T are a position observable E_ρ and a momentum observable F_ν, where

$$\mathrm{d}\rho(q) = e(q)\,\mathrm{d}q, \qquad e(q) = \sum_i \lambda_i|\varphi_i(-q)|^2, \tag{15}$$

$$\mathrm{d}\nu(p) = f(p)\,\mathrm{d}p, \qquad f(p) = \sum_i \lambda_i|\widehat{\varphi}_i(-p)|^2. \tag{16}$$

The form of ρ and ν in (15) and (16) implies that, in general, the margins E_ρ and F_ν do not determine G_T, that is, another covariant phase-space observable $G_{T'}$ may have the same margins. Indeed, the functions $|\varphi(\cdot)|$ and $|\widehat{\varphi}(\cdot)|$ do not define the vector φ uniquely up to a phase factor. (This is also known as the Pauli problem.)

Example 1. Consider the functions

$$\varphi_{a,b}(q) = \left(\frac{2a}{\pi}\right)^{1/4} \mathrm{e}^{-(a+\mathrm{i}b)q^2}, \tag{17}$$

with $a, b \in \mathbb{R}$ and $a > 0$. The Fourier transform of $\varphi_{a,b}$ is

$$\widehat{\varphi}_{a,b}(p) = \left(\frac{a}{2\pi(a^2+b^2)}\right)^{1/4} \exp\left(-\frac{ap^2}{4(a^2+b^2)}\right) \exp\left(\frac{\mathrm{i}bp^2}{4(a^2+b^2)} - \frac{\mathrm{i}}{2}\arctan\frac{b}{a}\right). \tag{18}$$

For $b \neq 0$, we see that $T_1 = |\varphi_{a,b}\rangle\langle\varphi_{a,b}|$ and $T_2 = |\varphi_{a,-b}\rangle\langle\varphi_{a,-b}|$ are different, but the margins of G_{T_1} and G_{T_2} are the same position and momentum observables E_ρ and F_ν, with

$$\mathrm{d}\rho(q) = \left(\frac{2a}{\pi}\right)^{1/2} \mathrm{e}^{-2aq^2}\,\mathrm{d}q, \qquad \mathrm{d}\nu(p) = \left(\frac{a}{2\pi(a^2+b^2)}\right)^{1/2} \exp\left(-\frac{ap^2}{2(a^2+b^2)}\right)\mathrm{d}p.$$

As ρ and ν in (15) and (16) arise from the same state T, a multitude of uncertainty relations can be derived for the observables E_ρ and F_ν. One of the most common uncertainty relation is in terms of variances. Namely, let $\mathrm{Var}(p)$ denote the variance of a probability measure p,

$$\mathrm{Var}(p) = \int \left(y - \int x\,\mathrm{d}p(x)\right)^2 \mathrm{d}p(y).$$

Then for any state S,

$$\mathrm{Var}\big(p_S^{E_\rho}\big)\mathrm{Var}\big(p_S^{F_\nu}\big) \geqslant 1. \tag{19}$$

(See, e.g., [7], section III.2.4 or [22], section 5.4.) The lower bound in (19) can be achieved only if

$$\mathrm{Var}(\rho)\,\mathrm{Var}(\nu) = \tfrac{1}{4}, \tag{20}$$

and it is well known that (20) holds if and only if $T = |\varphi\rangle\langle\varphi|$ and φ is a Gaussian function of the form

$$\varphi(q) = (2a/\pi)^{1/4}\,\mathrm{e}^{\mathrm{i}bq}\,\mathrm{e}^{-a(q-c)^2}, \qquad a > 0, \quad b, c \in \mathbb{R}. \tag{21}$$

It is also easily verified that choosing $S = T$ the equality in (19) is indeed obtained.

Proposition 7. *Let E_ρ be a position observable and F_ν a momentum observable. If E_ρ and F_ν have a joint observable, then they also have a joint observable which is a covariant phase-space observable.*

The proof of proposition 7 is given in the appendix.

Corollary 8. *A position observable E_ρ and a momentum observable F_ν are functionally coexistent if and only if there is a state $T \in \mathcal{S}(\mathcal{H})$ such that ρ and ν are given by (15) and (16). In particular, the uncertainty relation (19) is a necessary condition for functional coexistence, and thus, for the joint measurability of E_ρ and F_ν.*

Remark 9. As the canonical position observable and the canonical momentum observable are Fourier equivalent (see (2)), one may also want to require this connection from a fuzzy position observable E_ρ and a fuzzy momentum observable F_ν. This requirement, in general, simply leads to condition $\rho = \nu$. Let us consider the case when E_ρ and F_ν are the margins of the covariant phase-space observable G_T generated by a pure state $T = |\varphi\rangle\langle\varphi|$. Then $\rho = \nu$ exactly when $|\varphi| = |\widehat{\varphi}|$. To give an example when this condition is satisfied, suppose $\varphi = \varphi_{a,b}$, with $\varphi_{a,b}$ defined in equation (17). By equation (18), the condition $|\varphi_{a,b}| = |\widehat{\varphi}_{a,b}|$ is equivalent to

$$a^2 + b^2 = \tfrac{1}{4}. \tag{22}$$

Thus, if the numbers a and b are chosen so that they satisfy (22), the vector $\varphi_{a,b}$ defines Fourier equivalent position and momentum observables.

We end this section with an observation about a (lacking) localization property of a joint observable of position and momentum observables. We wish to emphasize that G in proposition 10 is not assumed to be a covariant phase-space observable.

Proposition 10. *Let G be a joint observable of a position observable E_ρ and a momentum observable F_ν and let $Z \in \mathcal{B}(\mathbb{R}^2)$ be a bounded set. Then*

(i) $\|G(Z)\| \neq 1$;
(ii) there exists a number $k_Z < 1$ such that for any $T \in \mathcal{S}(\mathcal{H})$, $p_T^G(Z) \leqslant k_Z$.

Proof.

(i) It follows from proposition 7 and the Paley–Wiener theorem that either ρ or ν has an unbounded support. Let us assume that, for instance, ρ has an unbounded support.
Let $Z \in \mathcal{B}(\mathbb{R}^2)$ be a bounded set. Then the closure $\bar{Z}$ is compact and also the set

$$X := \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} : (x, y) \in \bar{Z}\} \subset \mathbb{R}$$

is compact. Since

$$\|G(Z)\| \leqslant \|G(X \times \mathbb{R}\| = \|E_\rho(X)\| \tag{23}$$

and

$$\|E_\rho(X)\| = \text{ess sup}_{x\in\mathbb{R}}\rho(X - x) \leqslant \sup_{x\in\mathbb{R}}\rho(X - x), \tag{24}$$

it is enough to show that

$$\sup_{x\in\mathbb{R}}\rho(X - x) < 1. \tag{25}$$

Let us suppose, in contrast, that

$$\sup_{x\in\mathbb{R}}\rho(X - x) = 1. \tag{26}$$

This means that there exists a sequence $(x_n)_{n\geqslant 1} \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} \rho(X - x_n) = 1. \tag{27}$$

Since $\rho(\mathbb{R}) = 1$ and X is a bounded set, the sequence $(x_n)_{n\geqslant 1}$ is also bounded. It follows that $B := \bigcup_{n=1}^{\infty} X - x_n$ is a bounded set and by (27) we have $\rho(B) = 1$. This is in contradiction with the assumption that ρ has an unbounded support. Hence, (26) is false and (25) follows.

(ii) From (i) it follows that

$$1 > k_Z := \|G(Z)\| = \sup\{\langle\psi|G(Z)\psi\rangle \mid \psi \in \mathcal{H}, \|\psi\| = 1\}.$$

Let $T \in \mathcal{S}(\mathcal{H})$ and let $\sum_i \lambda_i|\psi_i\rangle\langle\psi_i|$ be the spectral decomposition of T. Then

$$p_T^G(Z) = \text{tr}[TG(Z)] = \sum_i \lambda_i\langle\psi_i|G(Z)\psi_i\rangle \leqslant k_Z.$$

□

5. Coexistence of position and momentum observables

Since coexistence is, *a priori*, a more general concept than functional coexistence, we are still left with the problem of characterizing coexistent pairs of position and momentum observables. In lack of a general result we close our investigation with some observations on this problem.

Proposition 11. *Let E_ρ be a position observable and F_ν a momentum observable. If* $\text{ran}(E_\rho) \cup \text{ran}(F_\nu)$ *contains a nontrivial projection (not equal to O or I), then E_ρ and F_ν are not coexistent.*

Proof. Let us assume, in contrast, that there exists an observable G such that $\text{ran}(E_\rho) \cup \text{ran}(F_\nu) \subseteq \text{ran}(G)$. Suppose, for instance, that $E_\rho(X)$ is a nontrivial projection. Then $E_\rho(X)$ commutes with all operators in the range of G (see, e.g., [23]). In particular, $E_\rho(X)$ commutes with all $F_\nu(Y)$, $Y \in \mathcal{B}(\mathbb{R})$. However, this is impossible by the result proved in [16] and [17] (see the beginning of the proof of proposition 4). □

Corollary 12. *Let E_ρ be a position observable which is a convex combination of two sharp position observables. Then E_ρ is not coexistent with any momentum observable F_ν.*

Proof. Let E_{ρ_1}, E_{ρ_2} be sharp position observables with $\rho_1 = \delta_a, \rho_2 = \delta_b$, and $E_\rho = tE_{\rho_1} + (1-t)\,E_{\rho_2}$ for some $0 \leqslant t \leqslant 1$. This means that $\rho = t\delta_a + (1-t)\,\delta_b$. If $a = b$, or $t \in \{0, 1\}$, then E_ρ is a sharp observable and the claim follows from proposition 11. Let us then assume that $a < b$ and $0 < t < 1$. Take $X = \bigcup_{n\in\mathbb{Z}} [n(b-a), (n+1/2)\,(b-a)]$. For any $x \in \mathbb{R}$, we have

$$\begin{aligned}\rho(X-x) &= t\delta_x(X-a) + (1-t)\delta_x(X-b)\\ &= t\sum_{n=-\infty}^{\infty}\delta_x\,([nb-(n+1)\,a, (n+1/2)\,b-(n+3/2)\,a])\\ &\quad + (1-t)\sum_{n=-\infty}^{\infty}\delta_x([(n-1)\,b-na, (n-1/2)\,b-(n+1/2)\,a])\\ &= \sum_{n=-\infty}^{\infty}\delta_x([nb-(n+1)\,a, (n+1/2)\,b-(n+3/2)\,a])\\ &= \delta_x(X-a).\end{aligned}$$

It follows that $E_\rho(X) = \Pi_Q(X-a)$. Since the projection $\Pi_Q(X-a)$ is nontrivial, the claim follows from proposition 11. $\square$

Evidently, corollary 12 has also a dual statement with the roles of position and momentum observables reversed.

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Appendix. Proof of proposition 7

In order to prove proposition 7 we need some general results about means on topological spaces, and for convenience they are briefly reviewed. The following material is based on [24], chapter IV, section 17, and [6].

Let Ω be a locally compact separable metric space with a metric d. By $BC(\Omega)$ we denote the Banach space of complex-valued bounded continuous functions on Ω, with the uniform norm $\|f\|_\infty = \sup_{x\in\Omega}|f(x)|$. The linear subspace of continuous functions with compact support is denoted by $C_c(\Omega)$. Adding the index ^{r} we denote the subsets of real functions in $BC(\Omega)$ or in $C_c(\Omega)$. With the index $^+$ we denote the subsets of positive functions.

Definition 13. *A mean on Ω is a linear functional*

$$m : BC(\Omega) \longrightarrow \mathbb{C}$$

such that:

(i) $m(f) \geqslant 0$ *if* $f \in BC^+(\Omega)$*;*
(ii) $m(1) = 1$.

For a mean m on Ω we denote

$$m(\infty) = 1 - \sup\big\{m(f)\big|f \in C_c^+(\Omega), f \leqslant 1\big\}.$$

Let m be a mean on Ω. By the Riesz representation theorem, there exists a unique positive Borel measure m_0 on Ω such that

$$m(f) = \int_\Omega f(x)\,\mathrm{d}m_0(x) \qquad \forall f \in C_c(\Omega).$$

By the inner regularity of m_0 we have

$$m_0(\Omega) = \sup\big\{m(f)\,\big|\, f \in C_c^+(\Omega),\ f \leqslant 1\big\} = 1 - m(\infty) \leqslant 1.$$

In particular, any function in $BC(\Omega)$ is integrable with respect to m_0. For any $f \in BC(\Omega)$, we use the abbreviated notation

$$m_0(f) := \int_\Omega f(x)\,\mathrm{d}m_0(x).$$

Proposition 14. *If* $m(\infty) = 0$, *then*

$$m(f) = m_0(f) \qquad \forall f \in BC(\Omega).$$

Proof. We fix a point $x_0 \in \Omega$. For all $R > 0$ we define

$$g_R(x) = \begin{cases} 1 & \text{if} \quad d(x_0, x) \leqslant R/2, \\ 3/2 - d(x_0, x)/R & \text{if} \quad R/2 < d(x_0, x) \leqslant 3R/2, \\ 0 & \text{if} \quad d(x_0, x) > 3R/2. \end{cases}$$

Then $g_R \in C_c^+(\Omega)$ and $g_R \leqslant 1$. Moreover, for any $f \in C_c^+(\Omega)$ such that $f \leqslant 1$ there exists $R > 0$ such that $f \leqslant g_R$, and hence

$$1 = \sup\big\{m(f)\,\big|\, f \in C_c^+(\Omega),\ f \leqslant 1\big\} = \lim_{R\to\infty} m(g_R).$$

Let $f \in BC^+(\Omega)$ and $R > 0$. Since $g_R f \in C_c(\Omega)$, we have

$$m(f) = m_0(g_R f) + m((1 - g_R) f). \tag{A.1}$$

We have $0 \leqslant g_R f \leqslant f$, f is m_0-integrable and $\lim_{R\to\infty} g_R(x) f(x) = f(x)$ for all $x \in \Omega$. Therefore, by the dominated convergence theorem we have

$$\lim_{R\to\infty} \int_\Omega g_R(x) f(x)\,\mathrm{d}m_0(x) = \int_\Omega f(x)\,\mathrm{d}m_0(x).$$

For the other term in sum (A.1), we have

$$m((1 - g_R) f) \leqslant \|f\|_\infty m(1 - g_R) \underset{R\to\infty}{\longrightarrow} \|f\|_\infty m(\infty) = 0.$$

Taking the limit $R \to \infty$ in (A.1) we then get

$$m(f) = m_0(f).$$

If $f \in BC(\Omega)$, we write $f = f_1 + \mathrm{i} f_2$ with $f_1, f_2 \in BC^r(\Omega)$, and $f_i = f_i^+ - f_i^-$ with $f_i^\pm = \frac{1}{2}(|f_i| \pm f_i) \in BC^+(\Omega)$, and we use the previous result to obtain the conclusion. $\square$

Let $i \in \{1, 2\}$. For each $f \in BC(\Omega)$ we define

$$\widetilde{f}_i(x_1, x_2) := f(x_i) \qquad \forall x_1, x_2 \in \Omega.$$

Clearly, $\widetilde{f}_i \in BC(\Omega \times \Omega)$. For a mean $m : BC(\Omega \times \Omega) \longrightarrow \mathbb{C}$, we then define

$$m_i(f) := m(\widetilde{f}_i) \qquad \forall f \in BC(\Omega).$$

The linear functional $m_i : BC(\Omega) \longrightarrow \mathbb{C}$ is a mean on Ω, which we call the ith *margin* of m.

Proposition 15. *Let m be a mean on $\Omega \times \Omega$. If* $m_1(\infty) = m_2(\infty) = 0$, *then* $m(\infty) = 0$.

Proof. For all $R > 0$, we define the function $g_R \in C_c(\Omega)$ as in the proof of proposition 14. We set

$$h_R(x_1, x_2) = g_R(x_1)g_R(x_2).$$

Clearly, $h_R \in C_c^+(\Omega \times \Omega)$, and, if $h \in C_c^+(\Omega \times \Omega)$ and $\leqslant 1$, there exists $R > 0$ such that $h \leqslant h_R$. Since

$$\begin{aligned} 1 - h_R(x_1, x_2) &= (1 - g_R(x_1)) + g_R(x_1)(1 - g_R(x_2)) \\ &\leqslant (1 - g_R(x_1)) + (1 - g_R(x_2)), \end{aligned}$$

we have

$$m(1 - h_R) \leqslant m_1(1 - g_R) + m_2(1 - g_R),$$

and the claim follows from

$$\begin{aligned} m(\infty) = 1 - \lim_{R\to\infty} m(h_R) &\leqslant \lim_{R\to\infty} m_1(1 - g_R) + \lim_{R\to\infty} m_2(1 - g_R) \\ &= m_1(\infty) + m_2(\infty) = 0. \end{aligned}$$

□

For a positive Borel measure m_0 on $\Omega \times \Omega$, we denote by $(m_0)_i$, $i = 1, 2$, the two measures on Ω which are margins of m_0.

Proposition 16. *Let m be a mean on $\Omega \times \Omega$. If $m(\infty) = 0$, then $(m_0)_i = (m_i)_0$ for $i = 1, 2$.*

Proof. Let $f \in C_c(\Omega)$. By proposition 14 we have

$$m_0(\widetilde{f}_i) = m(\widetilde{f}_i).$$

Using this equality and the definitions of $(m_0)_i$ and $(m_i)_0$ we get

$$(m_0)_i(f) = m_0(\widetilde{f}_i) = m(\widetilde{f}_i) = m_i(f) = (m_i)_0(f).$$

□

Definition 17. *An operator-valued mean on Ω is a linear mapping*

$$M : BC(\Omega) \longrightarrow \mathcal{L}(\mathcal{H})$$

such that:

(i) $M(f) \geqslant O$ *if* $f \in BC^+(\Omega)$*;*
(ii) $M(1) = I$.

For an operator-valued mean M on Ω we denote

$$M(\infty) = I - \mathrm{LUB}\big\{M(f) \,\big|\, f \in C_c^+(\Omega),\ f \leqslant 1\big\}.$$

The least upper bound in the above definition exists by virtue of proposition 1 in [25].

Let M be an operator-valued mean on Ω. For each $f \in BC^r(\Omega)$, we have

$$M(f - \|f\|_\infty 1) \leqslant O, \qquad M(f + \|f\|_\infty 1) \geqslant O.$$

It follows that

$$\|M(f)\| \leqslant \|f\|_\infty.$$

By theorem 19 in [25], there exists a unique positive operator measure M_0 on Ω such that

$$M(f) = \int_\Omega f(x)\,\mathrm{d}M_0(x) \qquad \forall f \in C_c(\Omega),$$

where the integral is understood in the weak sense. Similarly to the scalar case we have

$$M_0(\Omega) = I - M(\infty) \leqslant I, \tag{A.2}$$

and, for any $f \in BC(\Omega)$ we define

$$M_0(f) := \int_\Omega f(x)\,\mathrm{d}M_0(x).$$

Given an operator-valued mean M on Ω and a unit vector $\psi \in \mathcal{H}$, we set

$$m_\psi(f) := \langle\psi|M(f)\psi\rangle \qquad \forall f \in BC(\Omega).$$

It is clear that m_ψ is a mean on Ω. By proposition 1 in [25],

$$m_\psi(\infty) = \langle\psi|M(\infty)\psi\rangle.$$

Proposition 18. *If $M(\infty) = O$, then*

$$M(f) = M_0(f) \qquad \forall f \in BC(\Omega).$$

Proof. For a unit vector $\psi \in \mathcal{H}$ and a function $f \in C_c(\Omega)$, we have by definition

$$(m_\psi)_0(f) = \langle\psi|M_0(f)\psi\rangle,$$

and this equality is valid also for any $f \in BC(\Omega)$. Since

$$m_\psi(\infty) = \langle\psi|M(\infty)\psi\rangle = 0,$$

it follows from proposition 14 that the functional m_ψ on $BC(\Omega)$ coincides with integration with respect to the measure $(m_\psi)_0$. If $f \in BC(\Omega)$, we then have

$$\langle\psi|M_0(f)\psi\rangle = (m_\psi)_0(f) = m_\psi(f) = \langle\psi|M(f)\psi\rangle,$$

and the claim follows. □

The margins M_1 and M_2 of an operator-valued mean M on $\Omega \times \Omega$ are defined in an analogous way as in the case of scalar means.

Proposition 19. *Let M be an operator-valued mean on $\Omega \times \Omega$.*

(i) If $M_1(\infty) = M_2(\infty) = O$, then $M(\infty) = O$.
(ii) If $M(\infty) = O$, then $(M_0)_i = (M_i)_0$.

Proof.

(i) Let $\psi \in \mathcal{H}$ be a unit vector. We have, by definition, $(m_\psi)_i(f) = \langle\psi|M_i(f)\psi\rangle\ \forall f \in BC(\Omega)$ and $(m_\psi)_i(\infty) = \langle\psi|M_i(\infty)\psi\rangle$. It follows from proposition 15 that $m_\psi(\infty) = 0$. Since this is true for any unit vector, $M(\infty) = O$.
(ii) As in the scalar case, by proposition 18, we have

$$(M_0)_i(f) = M_0(\tilde{f}_i) = M(\tilde{f}_i) = M_i(f) = (M_i)_0(f).$$ □

With these results we are ready to prove proposition 7.

Proof of proposition 7. Given a function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ and $(q, p) \in \mathbb{R} \times \mathbb{R}$, we denote by $f^{(q,p)}$ the translate of f,

$$f^{(q,p)}(x, y) := f(x + q, y + p) \qquad \forall x, y \in \mathbb{R}.$$

Since $\mathbb{R} \times \mathbb{R}$ (with addition) is an Abelian group, there exists a mean m on $\mathbb{R} \times \mathbb{R}$ such that

$$m(f^{(q,p)}) = m(f)$$

for all $f \in BC(\mathbb{R} \times \mathbb{R})$ and $(q, p) \in \mathbb{R} \times \mathbb{R}$ (see [24], theorem IV.17.5).

Let M_0 be a joint observable of E_ρ and F_ν. For each $f \in BC(\mathbb{R}\times\mathbb{R})$, for all $\varphi, \psi \in \mathcal{H}$ and $q, p \in \mathbb{R}$ we define

$$\Theta[f;\varphi,\psi](q,p) := \langle W(q,p)^*\varphi | M_0(f^{(q,p)}) W(q,p)^*\psi\rangle.$$

Since

$$\|M_0(f^{(q,p)})\| \leqslant \|f^{(q,p)}\|_\infty = \|f\|_\infty$$

and $W(q,p)$ is a unitary operator, we have

$$|\Theta[f;\varphi,\psi](q,p)| \leqslant \|f\|_\infty \|\varphi\|\|\psi\|$$

and hence, $\Theta[f;\varphi,\psi]$ is a bounded function. We claim that $\Theta[f;\varphi,\psi]$ is continuous. Since

$$\Theta[f;\varphi,\psi](x+q, y+p) = \Theta[f^{(q,p)}; W(q,p)^*\varphi, W(q,p)^*\psi](x,y),$$

it is sufficient to check continuity at $(0,0)$. We have

$$\begin{aligned}|\Theta[f;\varphi,\psi](q,p) - \Theta[f;\varphi,\psi](0,0)| &\leqslant |\langle W(q,p)^*\varphi | M_0(f^{(q,p)})(W(q,p)^*\psi - \psi)\rangle| \\ &\quad + |\langle (W(q,p)^*\varphi - \varphi) | M_0(f^{(q,p)})\psi\rangle| + |\langle \varphi | M_0(f^{(q,p)} - f)\psi\rangle| \\ &\leqslant \|f\|_\infty(\|\varphi\|\|W(q,p)^*\psi - \psi\| + \|W(q,p)^*\varphi - \varphi\|\|\psi\|) \\ &\quad + |\langle \varphi | M_0(f^{(q,p)} - f)\psi\rangle|.\end{aligned}$$

As $(q,p) \to (0,0)$, the first two terms go to 0 by the strong continuity of W, and the third by the dominated convergence theorem. We have thus shown that $\Theta[f;\varphi,\psi] \in BC(\mathbb{R}\times\mathbb{R})$.

For each $f \in BC(\mathbb{R}\times\mathbb{R})$ we can then define a bounded linear operator $M^{av}(f)$ by

$$\langle \varphi | M^{av}(f)\psi\rangle := m(\Theta[f;\varphi,\psi]).$$

It is also immediately verified that the correspondence $M^{av} : BC(\mathbb{R}\times\mathbb{R}) \longrightarrow \mathcal{L}(\mathcal{H})$ is an operator-valued mean on $\mathbb{R}\times\mathbb{R}$, and a short calculation shows that

$$M^{av}(f^{(q,p)}) = W(q,p)^* M^{av}(f) W(q,p). \tag{A.3}$$

If $f \in BC(\mathbb{R})$ and $(q,p) \in \mathbb{R}\times\mathbb{R}$, we have

$$\begin{aligned}\Theta[\widetilde{f}_1;\varphi,\psi](q,p) &= \big\langle W(q,p)^*\varphi \big| M_0\big(\widetilde{f}_1^{(q,p)}\big) W(q,p)^*\psi\big\rangle \\ &= \langle W(q,p)^*\varphi | W(q,p)^* E_\rho(f) W(q,p) W(q,p)^*\psi\rangle \\ &= \langle \varphi | E_\rho(f)\psi\rangle.\end{aligned}$$

(In particular, $\Theta[\widetilde{f}_1;\varphi,\psi]$ is a constant function.) Similarly,

$$\Theta[\widetilde{f}_2;\varphi,\psi](q,p) = \langle \varphi | F_\nu(f)\psi\rangle.$$

It follows that

$$M_1^{av}(f) = E_\rho(f), \qquad M_2^{av}(f) = F_\nu(f).$$

Since $E_\rho(\mathbb{R}) = F_\nu(\mathbb{R}) = I$, (A.2) shows that

$$M_1^{av}(\infty) = M_2^{av}(\infty) = O.$$

This together with proposition 19 implies that $M_0^{av}(\mathbb{R}\times\mathbb{R}) = I$ and

$$\big(M_0^{av}\big)_1 = E_\rho, \qquad \big(M_0^{av}\big)_2 = F_\nu.$$

By (A.3) the observable M_0^{av} satisfies the covariance condition (11). $\square$

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